Balancing of a Planar Bouncing Object *

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Abstract

While most previous work in planning manipulation tasks relies on the assumption of quasi-static conditions, there can be situations where the quasi-static assumption may not hold, and the assumptions about the environment must be relaxed. This is true, for example, in a situation where objects are making and breaking contact at high enough velocities that contact dynamics play a significant effect in the motion of the colliding objects.

There has been some work studying models of collision, in particular for the design and analysis of systems with intermittent constraints, and for the design of juggling robots. Our work extends previous studies in planar juggling to the case of a polygonal object, using the model of rigid body impulsive collision. Simulations verify the results of a linearized analysis.

1 Motivation

Most strategies for planning manipulation tasks rely on the assumption of quasi-static mechanics in the analysis of the physical system. This constrains the plans to situations that are slow enought that contact dynamics can be neglected.

There are situations where these assumptions do not hold or when a model of the contact dynamics would be useful. In catching an already moving or accelerating object, for instance, the inertial properties of the object affect the motion which results from the applied forces of collision. Knowledge of the magnitudes as well as the direction of forces and velocities becomes important. Juggling and table tennis are two such domains that have been explored in robotics. Catching of tossed objects is a related task in which such knowledge is useful.

Another such domain is in the manipulation of objects by sliding on a frictional support surface. Much work has been done in the analysis of quasistatic pushing in the presence of friction ([13], [12], [14]). [8] shows that the motion of an object on a frictional support surface can be determined if the pressure distribution of the object is known. [12] and [14] analyze this situation when the pressure distribution is not known. The analysis of [8] implies that for large enough applied forces, the motion of the object is essentially given by the acceleration due to the applied forces. Since the impact model assumes that at the moment of collision the impact force dominates all other forces, the use of controlled collision can be useful in situations where the pressure distribution is not completely known, particularly if the support friction is low.

Some work has been done studying models of dynamic collision for use in robotic domains. [15] designed a dynamically stable hopping robot, modelling the bounce as a spring and damper system with perfectly inelastic collision. This system was further analyzed by [9]. [1] designed a ping-pong playing robot which used a simple model of point-mass collision to predict the motion of the ball after striking. [18] attempts to characterize the qualitative behavior change in the motion of objects upon collision. [19] simulates and analyzes systems with intermittent constraints, and uses models of those systems in planning manipulation tasks. [5], [6] analyse and design a planar puck juggling system. [16] extends this to the 3-D case. Our work continues those studies, extending the planar puck juggling work of [5], [6] to objects with extent and orientation. Our goal is provide solutions to manipulation tasks that lie beyond the quasi-static domain. Existing "manipulation primitives" consist primarily of quasi-static pushing, grasping, and placing. This work is an initial step in creating more general primitives, whose power lies in their ability to take advantage of dynamics.

2 The problem

We have a planar object on a frictionless inclined plane, pulled by the influence of gravity down to a movable "table", against which it bounces with coefficient of restitution e. The coefficient of friction between table and object is μ. The contact between the table and the object is a point contact, at a known point on the object. We assume an impulsive

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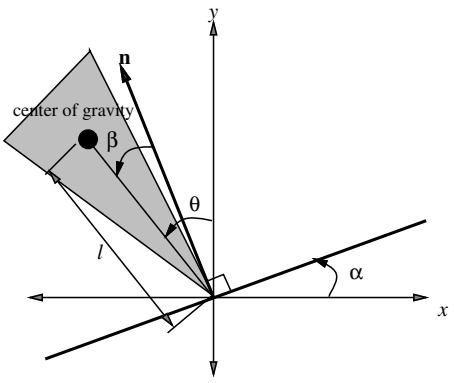

Figure 1: Basic System Geometry

impact model as described by [17] and used by [18], [19]. We also assume that at the moment of impact, the forces of collision dominate all other applied forces in determining the subsequent motion of the object.

The object is parameterized by $(x, y, \rho\theta)$, where (x, y) are the coordinates of the center of gravity of the object, θ is the orientation of the object, and ρ is the radius of gyration of the object ([7]). The desired orientation of the object will be set to $\theta = 0$. For simplicity, we can abstract away the dimensions of the actual object, and think of the object as a rod, whose center of gravity is the center of gravity of the object, located at length l from the contact point (figure 1). We want to have the object bouncing to a fixed height while maintaining the desired orientation. We would prefer the bouncing to occur at a fixed (impact) position in the $< x\, y >$ plane, but we will initially disregard this constraint, and study the simpler, lower dimensional unconstrained case.

In this examination, we will also assume perfect sensing and perfect control of the motion of the table, for the purpose of examining the question of whether the desired behavior is achievable in theory, before exploring problems in actual implementation.

2.1 The lossless case

We first look at the simplest case, the case $e = 1, \mu = 0$. In this case, the equation describing the change in velocity due to an impulsive collision can be written in terms of the preimpact velocity $\mathbf{v}^-$ as ([18])

$$\Delta\mathbf{v} = -2(\hat{\mathbf{n}}^T\mathbf{v}^-)\hat{\mathbf{n}}, \quad (1)$$

$$\hat{\mathbf{n}} = \frac{\rho}{\sqrt{\rho^2 + l^2\sin^2\beta}} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ \frac{l}{\rho}\sin\beta \end{bmatrix}. \quad (2)$$

As shown in Figure 1, α is the angle that the table makes with the horizontal in the counterclockwise direction, and β is the angle that the line from the contact point to the center of gravity makes with the table normal in the counterclockwise direction. Note that $\beta = \theta - \alpha$.

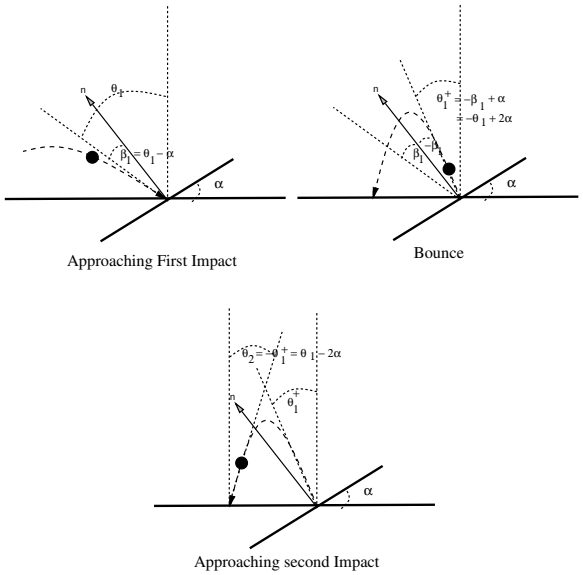

Figure 2: Ball bouncing against tilted table

Equation (1) simply says that in configuration space, the object upon impact will reverse its normal velocity component while its tangential component remains unchanged. Define the constraint surface to be the set of configurations for which the object touches the table without penetration for a given table orientation (see [11] for a discussion of configuration space). Then according to equation (1), the object will reflect about the normal to the constraint surface. This is an intuitive extension of the usual example of a perfectly elastic, frictionless point mass impact against a flat barrier.

To get an idea of possible solutions to our original problem, we look at the case of this point mass bouncing against our table. If the ball makes impact with a horizontal ($\alpha = 0$) table with a velocity vector at angle $\beta = \theta$ with the vertical, it will leave after impact with angle $-\theta$, and (if the table remains at the same height) strike the table again at angle θ. We would like to control the strikes so that θ eventually goes to zero, and the ball bounces straight up and down. One way to do this is tilt the table while simultaneously moving it so that impact always occurs at a given height in the vertical plane, say $y = 0$.

Suppose, as in figure 2, that the ball first makes impact with velocity at angle θ^- to the vertical. If the table is tilted at angle α at impact, the ball will leave the table at angle $-\beta = -(\theta^- - \alpha)$, which is equivalent to angle $-\theta^- + 2\alpha$ to the vertical. Hence, on the next impact, it will strike with the negative of that angle with respect to the vertical. This gives the system equation $\theta_{n+1} = \theta_n - 2\alpha_n$, where θ_n gives

the velocity angle with respect to the vertical just before the the nth impact. If we choose α to be proportional to θ, say $\alpha = \kappa\theta$, the recurrence relation becomes

$$\begin{aligned} \theta_n &= (1-2\kappa)\theta_{n-1} && (3)\\ &= (1-2\kappa)^n\theta_0. && (4)\end{aligned}$$

This will drive the system to its equilibrium state, $\theta = 0$, as long as $|1-2\kappa| < 1$, or $0 < \kappa < 1$. We now extend this strategy to the rigid body case.

2.1.1 System equations

We can rewrite equation 1 in the form

$$\mathbf{v}^+ = (\mathbf{I} - 2\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}^- \tag{5}$$

where $\mathbf{v}^-$ is the velocity vector immediately before impact, $\mathbf{v}^+$ is the velocity immediately after impact, and $\mathbf{I}$ is the identity matrix. After impact, during the ballistic phase, we have the equations

$$\begin{aligned} \dot{x}(t) &= \dot{x}_0 \\ \dot{y}(t) &= \dot{y}_0 - Gt \\ \rho\dot{\theta}(t) &= \rho\dot{\theta}_0 \end{aligned} \tag{6}$$

where the subscripted velocities are the velocities at the beginning of the ballistic phase, *i.e.* the velocities given by $\mathbf{v}^+$. G is the acceleration of gravity. The next impact occurs at $y_c = 0$, when the contact condition

$$l\cos\theta_0 + \dot{y}_0 t - \frac{1}{2}Gt^2 = l\cos(\theta_0 + \dot{\theta}_0 t) \tag{7}$$

is satisfied. The time until next impact is a function of θ_0, $\dot{\theta}_0$, and $\dot{y}_0$, which in turn are functions of the configuration just prior to impact, $(\theta^-, \mathbf{v}^-)$. Let the configuration at impact n be given by $\mathbf{x}_n = (\rho\theta, \dot{x}, \dot{y}, \rho\dot{\theta})_n$, and call the time of next impact $\tau(\mathbf{x}_n)$. Then the impact equations plus the contact condition describe a nonlinear recurrence relation, $\mathbf{x}_{n+1} = \mathbf{f}(\alpha, \mathbf{x}_n, \tau(\mathbf{x}_n))$.

If we set $\alpha = \kappa\theta$, then $\mathbf{f}$ is completely a function of $\mathbf{x}_n$, and we can try to find a fixed point $\mathbf{x}^*$. By inspection of (5), (6), and (7), we can see that $(\theta = \dot{\theta} = 0, \dot{y} = \dot{y}^*, \dot{x} = \dot{x}^*)$ defines a set of fixed points, with $\tau = -2\dot{y}^*/G$. Further, we know that for a lossless system the energy at contact, $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \rho^2\dot{\theta}^2) + mGl\cos\theta$, is constant from impact to impact, so our initial conditions determine the energy surface to which we are constrained. For specificity, we choose an equilibrium point from our set of possible fixed points to study. Since we would like our object to bounce in place horizontally, we choose $\dot{x}^* = 0$. Then, if we drop the object from a (center of gravity) height y_0, with an initial velocity vector $\mathbf{v}_0$, the corresponding value for $\dot{y}^*$ is given by

$$\dot{y}^* = -\sqrt{2G(y_0 - l\cos\theta) + \mathbf{v}_0{}^T\mathbf{v}_0}. \tag{8}$$

We wish to determine the stability of the state $\mathbf{x}^* = (0, 0, \dot{y}^*, 0)$: if we start off near, but not in this state, can we balance the object?

2.1.2 Linearized analysis

Given $\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*)$ for the recurrence relation $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, Taylor expansion about $\mathbf{x}^*$ gives us

$$\begin{aligned} \mathbf{f}(\mathbf{x}^* + \delta\mathbf{x}_n) &= \mathbf{x}^* + \delta\mathbf{x}_{n+1} \\ &= \mathbf{x}^* + \mathcal{J}(\mathbf{x}^*)\delta\mathbf{x}_n + h.o.t \end{aligned} \tag{9}$$

where $\mathcal{J}(\mathbf{x})$ is the Jacobian of $\mathbf{f}(\cdot)$ evaluated at $\mathbf{x}$. Ignoring higher order terms gives the approximate linear system

$$\begin{aligned} \delta\mathbf{x}_{n+1} &\approx \mathcal{J}(\mathbf{x}^*)\delta\mathbf{x}_n \\ &\approx [\mathcal{J}(\mathbf{x}^*)]^n\delta\mathbf{x}_0. \end{aligned} \tag{10}$$

The linearized system around $\mathbf{x}^*$ is given by the system matrix $\mathcal{J}$, which has four eigenvalues λ_i, two of which are unity, with eigenvectors $[0100]^T$ and $[0010]^T$, and the other two given by

$$\frac{2\dot{y}^{*2}l(1-\kappa) + G\rho^2}{G\rho^2} \pm \frac{2\dot{y}^* l\sqrt{\kappa-1}\sqrt{\dot{y}^{*2}l(\kappa-1) - G\rho^2}}{G\rho^2} \tag{11}$$

If all the eigenvectors $\mathbf{V}_i$ are distinct, then the system solution is given by

$$\delta\mathbf{x}_n = \sum_{i=1}^{4} c_i\mathbf{V}_i{\lambda_i}^n, \tag{12}$$

the c_i being functions of the initial conditions. In order for the solution to be stable, all the eigenvalues must be contained in the closed unit disc of the complex plane.

Because this system is conservative, $\mathcal{J}$ has determinant 1. It can be shown ([10], [2]) that the following are true in this case:

1. As with all systems where the entries of $\mathcal{J}$ are real, if λ is an eigenvalue of the system, then λ^*, the complex conjugate, is also an eigenvalue.

2. If λ is an eigenvalue of the system, then $1/\lambda$ is also an eigenvalue.

Therefore, for a conservative system, stability is only possible if the linear system values are all on the unit circle. The case where the eigenvalues are ± 1 more problematic

[2], but fortunately in the case at hand, the two complex eigenvalues seem to be the critical values to examine.

Analyzing $\lambda_{3,4}$ for different values of κ gives

$$\kappa < 1: \quad |\lambda_3| > 1, |\lambda_4| < 1 \quad \lambda_{3,4} \text{ real} \tag{13}$$

$$1 < \kappa < \kappa_{crit}: \quad |\lambda_{3,4}| = 1 \quad \lambda_{3,4} \text{ complex} \tag{14}$$

$$\kappa > \kappa_{crit}: \quad |\lambda_3| < 1, |\lambda_4| > 1 \quad \lambda_{3,4} \text{ real} \tag{15}$$

$$\kappa_{crit} = 1 + \tfrac{G\rho^2}{\dot{y}^{*2} l}. \tag{16}$$

The boundaries of the region given by (14) are special cases, because for those values of κ not all the eigenvectors are distinct, and the analysis is more complicated. For the region of stable κ, the eigenvectors corresponding to $\lambda_{3,4}$ are complex conjugate, and (12) can be written in the form

$$\delta \mathbf{x}_n = \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ 0 \end{bmatrix} + 2\,\mathrm{Re}\left(c_3\, \mathrm{e}^{\mathrm{j}\,\sigma n} \begin{bmatrix} a + \mathrm{j}\,b \\ g \\ 0 \\ h \end{bmatrix} \right), \qquad \sigma, a, b, g, h \in \Re \tag{17}$$

and the initial conditions give for the constants of proportionality:

$$c_1 = \delta\dot{x}_0 - \frac{\rho\delta\dot{\theta}_0}{h} g, \tag{18}$$

$$c_2 = \delta\dot{y}_0, \tag{19}$$

$$c_3 = \frac{\rho\delta\dot{\theta}_0}{2h} + \mathrm{j}\left(\frac{a\rho\delta\dot{\theta}_0}{2bh} - \frac{\rho\delta\theta_0}{2b}\right). \tag{20}$$

The linear analysis predicts that in the neighborhood about the equilibrium point, if there is any deviation from $\dot{y}^*$, it will stay constant; if $\delta\dot{x}_0 = 0$ and $\delta\dot{\theta}_0 = 0$, then $\delta\dot{x}, \delta\theta$, and $\delta\dot{\theta}$ will all oscillate about the origin at frequency σ, with amplitudes determined by $\delta\theta_0$. These, then, are the initial conditions that determine the stability of $\mathbf{x}^*$. If either $\delta\dot{x}_0$ or $\delta\dot{\theta}_0$ are nonzero, there will be a net x velocity, and the object will remain balanced, but travel horizontally as it bounces. Strictly speaking, this is not stable, since x (which we have been ignoring up until now) can increase without bound. But since $\mathbf{x}^*$ does not contain x, it does remain bounded. For the case $\kappa = \kappa_{crit} = 1 + (G\rho^2)/(\dot{y}^{*2} l)$, linear analysis predicts instability of the system.

2.2 Extensions to more general cases

2.2.1 Inelastic collision ($e \neq 1$)

If we assume that $mass_{table} \gg mass_{object}$, then

$$\mathbf{v}^+ = (\mathbf{I} - (1+e)\hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}^- + (1+e)\hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{v}_{table} \tag{21}$$

([18]) and $\mathbf{v}_{table}$ remains unchanged due to our assumption about the relative masses. Setting

$$\mathbf{v}_{table} = -\frac{1-e}{1+e}\mathbf{v}^-, \tag{22}$$

reduces (21) to the equation for the lossless case. Note that the contact velocities are given by

$$\begin{aligned} \dot{x}_c &= \dot{x} + l\dot{\theta}\cos\theta \\ \dot{y}_c &= \dot{y} + l\dot{\theta}\sin\theta, \end{aligned} \tag{23}$$

when measured in the global frame (figure (1)). Then another choice for $\mathbf{v}_{table}$ that will give the same system equations as (22) is

$$\mathbf{v}_{table} = -\frac{1-e}{1+e}\hat{\mathbf{n}}\hat{\mathbf{n}}^T \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ 0 \end{bmatrix} \tag{24}$$

If in addition to having the object bounce straight up and down, one also wanted the object to bounce to a specific (center of gravity) height, y^*, the associated (unit mass) energy level $\eta^* = Gy^*$ can be used as an additional feedback term. As in [6], we can use the feedback law

$$\begin{aligned} \mathbf{v}_{table} &= -\left[\tfrac{1-e}{1+e} + \kappa_E(\eta^* - \eta)\right]\mathbf{v}_c, \\ \eta &= Gl\cos\theta + \tfrac{1}{2}\dot{y}^2 \qquad \text{at impact.} \end{aligned} \tag{25}$$

Simplification of the above equation for the point mass case, linearized about $\dot{y}^{*2}$, gives the approximation that for stability, κ_E should be in the range $0 < \kappa_E < \frac{4}{(1+e)\dot{y}^{*2}} \doteq \kappa_{Emax}$. α still is set to $\kappa\theta$, and the new critical value for κ is now

$$\kappa_{crit} = 1 + \frac{2G\rho^2}{(1+e)\dot{y}^{*2} l}. \tag{26}$$

The eigenvectors and eigenvalues of the linearized system are essentially the same (for $0 < \kappa < \kappa_{crit}$), except the eigenvalue corresponding to the eigenvector $[0010]^T$ (which is the eigenvector corresponding to $\delta\dot{y}$) is now given by the value $(1 - \frac{\kappa_E}{\kappa_{Emax}}) < 1$, reflecting the linearized prediction that the deviation in $\dot{y}$ goes to zero, *i.e.* that the system will converge to the correct energy surface.

2.2.2 Friction

If we remove the assumption that $\mu_{table} = 0$, the impact equations become nonlinear, reflecting the nonlinearity of Coulomb friction. Although the analytic approach becomes more difficult, empirical studies for various values of μ found this case to be unstable when using the table tilt rule explained above, somewhat contrary to expectations. Apparently, the law cannot compensate for the energy lost in the tangential direction, and in fact often added an impulse in a direction that increased θ, contributing to the tip-over of the object. This can be seen by looking at the equation for the moment due to the impulse (figure (3))

$$\begin{aligned} M &= P_n l\sin\beta + P_t l\cos\beta \\ &= P_n l\sin\beta + \mu P_n \,\mathrm{sgn}(-v_{tc}) l\cos\beta \end{aligned} \tag{27}$$

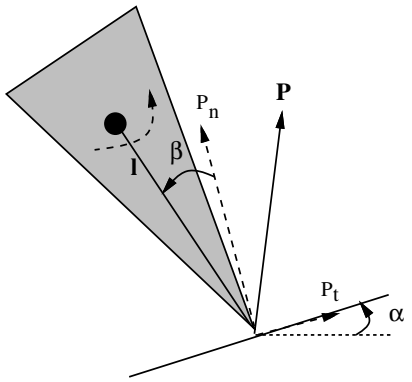

Figure 3: Tangential and Normal Impulses

for the case of sliding contact. Here P_t, P_n are the tangential and normal components of the contact, and $v_t c$ is the tangential contact velocity. For small angles, $\sin\beta \approx \beta$ and $\cos\beta \approx 1 - \beta^2 >> \beta$, so when β is smaller than μ, the moment due to frictional forces can potentially cancel out the desired moment, and cause the object to rotate in the wrong direction. This problem can be compensated for by increasing κ, which in general causes β to be larger. Increasing κ_E also prolongs the time that the object can be kept upright, but the tangential forces increase the horizontal motion of the object, and hence the energy dissipated to friction, and eventually the object falls over. This difficulty can probably be circumvented by striking the object at an angle that minimizes tangential velocity, or by striking at a different place.

2.3 Empirical verification

Simulation showed that the linear approximation predicted the stability of the system reasonably well for different choices of κ: the region described by (14) was stable for small initial velocities and for angular deviations up to about ± 0.4 radians ($\approx 23°$). When the initial impact angle was small, about ± 0.15 radians ($\approx 8.6°$) or less, the system was stable $\kappa = \kappa_{crit}$ inclusive. When the initial impact angle was in the range $\pm$(0.15 to 0.4) radians, κ had to be closer to unity for stability. The $\kappa = 1$ case is always unstable, but the $\kappa = \kappa_{crit}$ case can be stable. Note for comparison that the range ± 0.5 radians ($\approx 30°$), is the range over which the linear approximation $\sin x \approx x$ holds.

The examples shown are for simulations of an isosceles triangle of uniform mass distribution, 10 cm. wide at the base, and 20 cm. high. The radius of gyration about the center of gravity for this triangle is about 9.428 cm. The triangle was dropped with zero initial velocity from a height of 20 cm., with $\theta_0 = 0.3$ radians. The desired bouncing height was 25 cm. κ was set to 1.1 ($\kappa_{crit} = 1.5$), and κ_E to 0.165 ($\kappa_{Emax} = 1.65$).

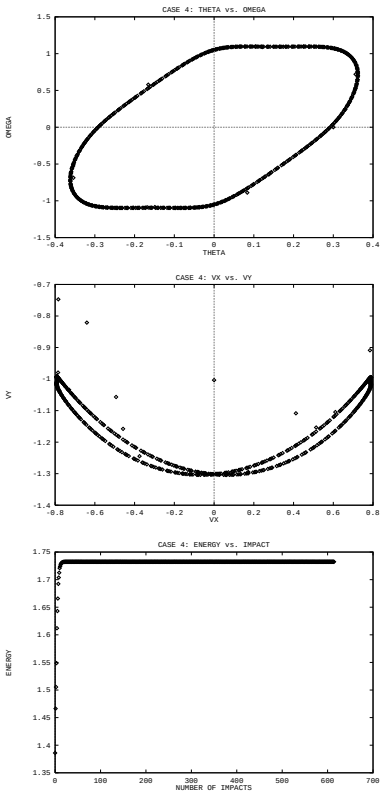

Figure 4: Example without noise: $\theta_0 = 0.3$ rad., $e = 0.5, \kappa = 1.1, \kappa_E = 0.165, \eta^* = 1.73$

The figures show projections of the system orbit in the $\theta\dot{\theta}$ plane (θ is on the horizontal axis) and the projections in the $\dot{x}\dot{y}$ plane. Notice that the $\theta\ \dot{\theta}$ projection is centered about the origin in both cases, showing that θ and $\dot{\theta}$ oscillate about zero. With no initial dropping velocities, $\dot{x}$ is also centered about the origin, and $\dot{y}$ is bounded in the neighborhood of $\dot{y}_0$, the y-velocity at first impact.

Experiments confirm that for low values of κ_E (about $0.1\kappa_{Emax}$, as in the examples shown) the energy converges to the correct level, and then the behavior of the system is similar to the lossless case. For values of κ_E much higher than $0.1\kappa_{Emax}$, the system is generally unstable.

Although the system has only been simulated, not built, an attempt was made to approximate imperfections in sensing and control by adding some Gaussian noise to the calculation of impact time used by the table. This changes the angle and velocity of the table at impact time, as well as the y-height at which contact is made. Zero mean gaussian noise with a standard deviation of 10 ms., which is about an order of magnitude less than the time between collisions, was used. Although the motion of the object with noise added is less smooth, the system remains stable in a similar range of initial conditions as the noiseless case.

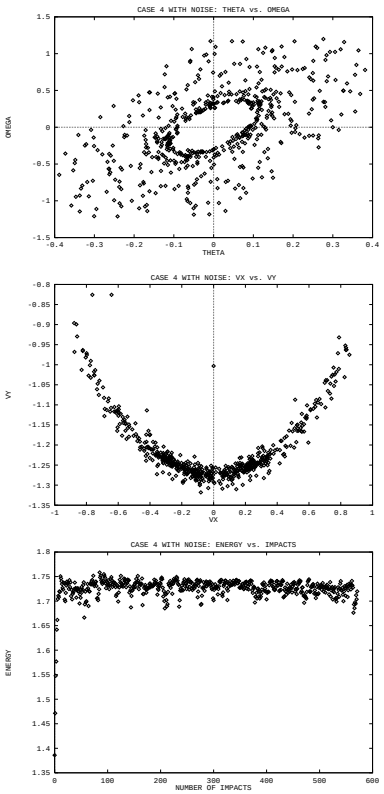

Figure 5: Same example with noise

3 Conclusions

The experiments show that, under ideal conditions, control of the object is theoretically possible using knowledge of the collision parameters. The biggest problems are, of course, dealing with friction and setting up the necessary sensing and control. Although indications are that stability can be maintained for reasonably accurate robot and sensors, handling friction will be necessary before this scheme can be considered feasible.

In terms of applicability to other domains, the results may also be useful in the dual problem of planning the acquisition of a stationary object with an accelerating hand, or in catching. For these tasks, we consider the desired stable state to be zero relative velocity (and distance) between the object and the robot hand. Then we would like to plan the movements of the hand so that the object does not fall or bounce away from the hand, but instead eventually settles there. Work on this is underway.

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